Countability 2

Let's prove $\mathbb{P}(\mathbb{N})$ is not countable.

- if we can index a set by $\mathbb{N}$, it is countable
  - ↳ if it is impossible to index that set by $\mathbb{N}$, it is uncountable

representation: $\mathbb{N} = \{0, 1, 2, 3 \ldots\}$

each element of $\mathbb{P}(\mathbb{N})$ can be represented by an infinitely long bit string: e.g., $0110\ldots$

$A = \{1, 4, 5\}$

$\mathbb{P}(A) = \{\emptyset, \{1\}, \{4\}, \{5\}, \{1,4\}, \{1,5\}, \{4,5\}, \{1,4,5\}\}$

$\emptyset$	$\{1\}$	$\{4\}$	$\{5\}$	$\{1,4\}$	$\{1,5\}$	$\{4,5\}$	$\{1,4,5\}$
000	100	010	001	110	101	011	111

if I claim $\mathbb{P}(\mathbb{N})$ is countable, I should be able to index it by $\mathbb{N}$ (write it in an ordered list)

$$
\left[\begin{array}{cccccccc}
\mathbb{N} & & b_0 & b_1 & b_2 & b_3 & b_4 & \\
0 & \rightarrow & 0 & 1 & 1 & 0 & 1 & \cdots \\
1 & \rightarrow & 1 & 0 & 1 & 1 & 1 & \cdots \\
2 & \rightarrow & 1 & 0 & 1 & 1 & 0 & \cdots \\
3 & \rightarrow & 0 & 1 & 1 & 0 & 0 & \cdots \\
4 & \rightarrow & 1 & 1 & 1 & 0 & 0 & \cdots \\
\vdots & & & & & & &
\end{array}\right.
$$

→ set of infinitely long bit vectors is uncountable

↓ can use this result

$$
\left[\begin{array}{c}
|\mathbb{P}(\mathbb{N})| > |\mathbb{N}| \\
|\mathbb{P}(A)| > |A|
\end{array}\right]
$$

new bit string: $\underline{1}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{1}\ \ldots$

it is impossible to index the list by $\mathbb{N}$ (build a bijection) because we can always create a new bit string not in the list.

$\mathbb{P}(\mathbb{N})$ is not countable.

Show that set of $\mathbb{R} \in [0,1)$ is uncountable

$\mathbb{R}$ are also uncountable

$\mathbb{N}$	
0	.1 0 0 0 . . .
1	.1 3 4 5 . . .
2	.3 9 7 4 . . .
3	.2 4 1 2 . . .
4	.9 9 9 9 . . .

new $\mathbb{R}$: .2 4 8 3

uncomputability and halting problem